

# Fermi co-ordinates and relativistic effects in non-inertial frames

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## Abstract

Fermi co-ordinates are proper co-ordinates of a local observer determined by his trajectory in space-time. Two observers at different positions belong to different Fermi frames even if there is no relative motion between them. Use of Fermi co-ordinates leads to several physical conclusions related to relativistic effects seen by observers in arbitrary motion. In flat space-time, the relativistic length seen by an observer depends only on his instantaneous velocity, not on his acceleration or rotation. In arbitrary space-time, for any observer the velocity of light is isotropic and equal to  $c$ , provided that it is measured by propagating a light beam in a small neighbourhood of the observer. The value of a covariant field measured at the position of the observer depends only on his instantaneous position and velocity, not on his acceleration. The notion of radiation is observer independent. A “freely” falling charge in curved space-time does not move along a geodesic and therefore radiates.

## 1 Introduction

In this paper I review some recent results which have led to progress in the understanding of relativistic effects seen by local observers arbitrarily moving in flat or curved space-time. Some technical details given in the cited papers are omitted here, but some conceptual details are explained in a slightly different and perhaps more illuminating way. I discuss only classical effects, not quantum effects, because quantum physics in non-inertial frames and curved space-time is not yet completely understood.

The proper co-ordinates of an observer determined by his trajectory in space-time are the so-called Fermi co-ordinates and they should be used in order to describe physical effects seen by him. The main point of this paper is the fact that two observers at different positions belong to different Fermi frames even if there is no relative motion between them. As I discuss later, this fact was not recognized in many previous papers, which led to several

misinterpretations and paradoxes. As I show in the paper, the correct interpretation of Fermi co-ordinates leads to several results which demonstrate that acceleration and rotation are less important than it was previously thought, because it appears that for many instantaneous physical effects only the instantaneous position and velocity are relevant.

## 2 Fermi co-ordinates

In physics, all dynamical equations of motion are certain differential equations that describe certain quantities as functions of space-time points. Space-time points are parametrized by their co-ordinates. It is convenient to write the equations of motion (as well as other related equations) in a form which is manifestly covariant with respect to general co-ordinate transformations. When one solves the equations, one must use some specific co-ordinates. The covariance provides that one can use any co-ordinates he wants, because later he can easily transform the results to any other co-ordinates. Therefore, it is convenient to choose co-ordinates that simplify the technicalities of the physical problem considered.

The general covariance is often interpreted as a statement that “physics does not depend on the co-ordinates chosen”. However, this is not so. The choice of co-ordinates is more than a matter of convenience. The main purpose of theoretical physics is to predict what will be *observed* under given circumstances. The main lesson we have learned from Lorentz co-ordinates is the fact that what an observer observes (time intervals, space intervals, components of a tensor, ...) depends on how the observer moves. Lorentz co-ordinates are proper co-ordinates of an observer that moves inertially in flat space-time. Fermi co-ordinates are the generalization of Lorentz co-ordinates to arbitrary motion in arbitrary space-time. If one is interested in how a physical system looks like to a specific observer, one must transform the results to the corresponding Fermi co-ordinates.

Fermi co-ordinates are determined by the (time-like) trajectory of the observer, by the rotation of the observer with respect to a local inertial observer and by the properties of space-time itself. The general geometrical construction of Fermi co-ordinates is well established [1]. Here I present the most important properties of Fermi co-ordinates:

1. Fermi co-ordinates are chosen such that the position of the observer is given by  $x^\mu = (t, 0, 0, 0)$ , where  $t$  is the time measured by a clock co-moving with the observer.
2. The metric expressed in Fermi co-ordinates has the property
$$g_{\mu\nu}(t, 0, 0, 0) = \eta_{\mu\nu} . \tag{1}$$
3. The connections  $\Gamma_{\beta\gamma}^\alpha$  vanish at  $(t, 0, 0, 0)$  if and only if the trajectory is a geodesic and there is no rotation.

The general geometrical construction of Fermi co-ordinates is not very useful for practical calculations. However, in flat space-time, Fermi co-ordinates can be constructed in an alternative way, more useful for practical calculations [2]. Here I present a very elegant form of this construction [3].

Let  $S$  be a Lorentz frame and let  $S'$  be the Fermi frame of the observer whose 3-velocity is  $u^i(t') \equiv \mathbf{u}(t')$ , as seen by an observer in  $S$ . In general,  $S'$  can also rotate, which can be

described by the rotation matrix  $A_{ji}(t') = -A_j^i(t')$ . The co-ordinate transformation between these two frames is given by

$$x^\mu = \int_0^{t'} f_0^\mu(t', 0; \mathbf{u}(t')) dt' + \int_C f_i^\mu(t', \mathbf{x}'; \mathbf{u}(t')) dx'^i, \quad (2)$$

where

$$f_\nu^\mu = \left( \frac{\partial f^\mu}{\partial x'^\nu} \right)_{\mathbf{u}=\text{const}}, \quad (3)$$

and

$$x^\mu = f^\mu(t', \mathbf{x}'; \mathbf{u}) \quad (4)$$

denotes the ordinary Lorentz transformation, i.e. the transformation between two Lorentz frames specified by the constant relative velocity  $\mathbf{u}$ . Explicitly,

$$\begin{aligned} f_0^0 &= \gamma, & f_j^0 &= -\gamma u_j, & f_0^i &= \gamma u^i, \\ f_j^i &= \delta_j^i + \frac{1-\gamma}{\mathbf{u}^2} u^i u_j, \end{aligned} \quad (5)$$

where  $u^j = -u_j$ ,  $\mathbf{u}^2 = u^i u_i$ ,  $\gamma = 1/\sqrt{1-\mathbf{u}^2}$  are evaluated at  $t'$ . In (2),  $C$  is an arbitrary curve with constant  $t'$ , starting from 0 and ending at  $-A_j^i(t')x'^j$ . The rotation matrix satisfies the differential equation

$$\frac{dA_{ij}}{dt} = -A_i^k \omega_{kj}, \quad (6)$$

where  $\omega_{ik} = \varepsilon_{ikl} \omega^l$ ,  $\varepsilon_{123} = 1$  and  $\omega^i(t')$  is the angular velocity as seen by an observer in  $S$ . The Fermi co-ordinates  $x'^\mu$  are constructed such that the space origins of  $S$  and  $S'$  coincide at  $t = t' = 0$ .

The metric tensor in  $S'$  is

$$\begin{aligned} g'_{ij} &= -\delta_{ij}, & g'_{0j} &= -(\boldsymbol{\omega}' \times \mathbf{x}')_j, \\ g'_{00} &= c^2 \left( 1 + \frac{\mathbf{a}' \cdot \mathbf{x}'}{c^2} \right)^2 - (\boldsymbol{\omega}' \times \mathbf{x}')^2, \end{aligned} \quad (7)$$

where

$$\omega'^i = \gamma(\omega^i - \Omega^i), \quad a'^i = \gamma^2 \left[ a^i + \frac{1}{\mathbf{u}^2} (\gamma - 1)(\mathbf{u} \cdot \mathbf{a}) u^i \right], \quad (8)$$

$\Omega^i$  is the time-dependent Thomas precession frequency

$$\Omega_i = \frac{1}{2\mathbf{u}^2} (\gamma - 1) \varepsilon_{ikj} (u^k a^j - u^j a^k), \quad (9)$$

and  $a^i = du^i/dt$  is the time-dependent acceleration.

In general space-time, instead of a closed formula (2), one can find the transformation between two Fermi frames in the form

$$x^\mu = f_\nu^\mu x'^\nu + f_{\nu\alpha}^\mu x'^\nu x'^\alpha + \dots, \quad (10)$$

where it is assumed that the two observers have the same position at  $t = t' = 0$ . One can show [4] that the quantity

$$f^\mu_\nu = \left( \frac{\partial x^\mu}{\partial x'^\nu} \right)_{x=x'=0} \quad (11)$$

is again given by (5), where  $u^i = dx^i/dt$  is the velocity of the observer in  $S'$  as seen by the observer in  $S$ , at the instant when the two observers have the same position.

From Property 2 we see that the space co-ordinates  $x^i$  are a generalization of Cartesian co-ordinates. However, this does not imply that an observer is not allowed to use polar co-ordinates, for example. The most general co-ordinate transformations that correspond merely to a redefinition of the co-ordinates of the same physical observer are the so-called restricted internal transformations [3], i.e. the transformations of the form

$$t' = f^0(t), \quad x'^i = f^i(x^1, x^2, x^3), \quad (12)$$

where  $t, x^i$  are Fermi co-ordinates. The quantities  $g_{00}dt^2$  and

$$dl^2 = -g_{ij}dx^i dx^j \quad (13)$$

do not change under restricted internal transformations. In order to describe physical effects as seen by a local observer, one must use Fermi co-ordinates modulo restricted internal transformations. For simplicity, in the rest of the paper I use Fermi co-ordinates.

Two observers with different trajectories have different Fermi frames. In particular, this implies that *even if there is no relative motion between two observers, they belong to different frames if they do not have the same position*. As we shall see, this fact was not realized in many previous papers, which led to many misinterpretations. At first sight, this fact contradicts the well-known fact that two inertial observers in flat space-time may be regarded as belonging to the same Lorentz frame if there is no relative motion between them. However, this is because their Fermi frames (with the space origins at their positions) are related by a translation of the space origin, which is a restricted internal transformation. In general, for practical purposes, two observers can be regarded as belonging to the same Fermi frame if there is no relative motion between them and the other observer is close enough to the first one, in the sense that the metric expressed in Fermi co-ordinates of the first observer does not depart significantly from  $\eta_{\mu\nu}$  at the position of the second one. It is an exclusive property of Minkowski co-ordinates, among other Fermi co-ordinates in flat or curved space-time, that the metric is equal to  $\eta_{\mu\nu}$  *everywhere*. This implies that two observers at different positions but with zero relative velocity may be regarded as belonging to the same co-ordinate frame only if they move inertially in flat space-time.

### 3 Relativistic contraction and the rate of clocks

In this section I study relativistic contraction and the rate of clocks as seen by an observer in flat space-time. As seen in the preceding section, the explicit co-ordinate transformation that determines the relevant Fermi frame is available for flat space-time.

For motivation, let us first review the problems [3] of the standard resolution [5, 6] of the Ehrenfest paradox. We study a rotating ring in a rigid non-rotating circular gutter with radius  $r$ . One introduces the co-ordinates of the rotating frame  $S'$

$$\varphi' = \varphi - \omega t, \quad r' = r, \quad z' = z, \quad t' = t, \quad (14)$$

where  $\varphi, r, z, t$  are cylindrical co-ordinates of the inertial frame  $S$  and  $\omega$  is the angular velocity. The metric in  $S'$  is given by

$$ds^2 = (c^2 - \omega^2 r'^2) dt'^2 - 2\omega r'^2 d\varphi' dt' - dr'^2 - r'^2 d\varphi'^2 - dz'^2. \quad (15)$$

It is generally accepted that the space line element should be calculated by the formula [7]

$$dl'^2 = \gamma'_{ij} dx'^i dx'^j, \quad i, j = 1, 2, 3, \quad (16)$$

where

$$\gamma'_{ij} = \frac{g'_{0i} g'_{0j}}{g'_{00}} - g'_{ij}. \quad (17)$$

This leads to the circumference of the ring

$$L' = \int_0^{2\pi} \frac{r' d\varphi'}{\sqrt{1 - \omega^2 r'^2/c^2}} = \frac{2\pi r'}{\sqrt{1 - \omega^2 r'^2/c^2}} \equiv \gamma(r') 2\pi r'. \quad (18)$$

The circumference of the same ring as seen from  $S$  is  $L = 2\pi r = 2\pi r'$ . Since the ring is constrained to have the same radius  $r$  as the same ring when it does not rotate,  $L$  is not changed by the rotation, but the proper circumference  $L'$  is larger than the proper circumference of the non-rotating ring. This implies that there are tensile stresses in the rotating ring. The problem is that it is assumed here that (14) defines the proper frame of the whole ring. This implies that an observer on the ring sees that the circumference is  $L' = \gamma L$ . The circumference of the gutter seen by him cannot be different from the circumference of the ring seen by him, so the observer on the ring sees that the circumference of the relatively moving gutter is *larger* than the proper circumference of the gutter, whereas we expect that he should see that it is smaller.

This problem resolves when one realizes that (14) does not define the proper frame of the ring. Each point on the ring belongs to a different Fermi frame. The co-ordinates (14) are actually Fermi co-ordinates (modulo a restricted internal transformation) of an observer that rotates in the centre of the ring. However, this raises another problem. If (16) is the correct definition of the space line element, then the observer that rotates in the centre should see that the circumference of the gutter is larger than the proper circumference of the gutter by a factor  $\gamma(r')$ . However,  $\omega r'/c$  can be arbitrarily large, so  $\gamma(r')$  can be not only arbitrarily large, but also even imaginary. On the other hand, we know from everyday experience that the apparent velocity  $\omega r'$  of stars, due to our rotation, can exceed the velocity of light, but we see neither a contraction nor an elongation of the stars observed. This implies that the definition of the space line element (16) is not always appropriate. In [7], (16) is derived by defining the space line element through a measuring procedure that lasts a finite time, so, in general, this formula is not appropriate for a definition of the *instantaneous* length. In flat

space-time, as shown in [3], if physics is described by Fermi co-ordinates modulo restricted internal transformations, a more appropriate definition of the space line element is (13).

Using the co-ordinate transformation (2), one can study the relativistic contraction in the same way as in the conventional approach with Lorentz frames. One assumes that two ends of a body are seen to have the same time co-ordinate. From (2) and (5) one can easily see that the co-ordinate transformation is linear in  $x'^i$ . As demonstrated in more detail in [3, 9], it implies that *an arbitrarily accelerated and rotating observer sees equal lengths of other differently moving objects as an inertial observer whose instantaneous position and velocity are equal to that of the arbitrarily accelerated and rotating observer.*

Using (2), one can also study the rate of clocks as seen by various observers. In particular, one can study the twin paradox for various motions of the observers. However, it is more interesting to study not only the time shift after the two differently moving observers eventually meet, but also the continuous changes of the time shifts during the motion. As can be seen from (2), it is the time dependence (not the space dependence) of the co-ordinate transformation that significantly differs from the ordinary Lorentz transformations. One cannot invert (2) simply by putting  $u^i \rightarrow -u^i$ . Therefore, inertial and non-inertial observers see quite different continuous changes of the time shift.

Here I present the results for uniform circular motion, derived in [3]. Assume that there are three clocks. One is at rest in  $S$ , so it moves inertially. The other two are moving around a circle with the radius  $R$  and have the velocity  $\omega R$  in the counter-clockwise direction, as seen by the observer in  $S$ . The relative angular distance between these two non-inertial clocks is  $\Delta\varphi_0$ , as seen in  $S$ . The inertial observer will see that the two non-inertial clocks lapse equally fast, so I choose that he sees that they show the same time. He will see the clock rate  $t = \gamma t'$ , where  $\gamma = 1/\sqrt{1 - \omega^2 R^2/c^2}$ . The observer co-moving with one of the non-inertial clocks will not see that the other non-inertial clock shows the same time as his clock. He will see the constant time shift given by the equation

$$\gamma\omega(t'' - t') = \beta^2 \sin(\gamma\omega(t'' - t') + \Delta\varphi_0) , \quad (19)$$

where  $\beta^2 \equiv \omega^2 R^2/c^2$  and  $t''$  is the time of the other non-inertial clock. Finally, let us see how the inertial clock looks like from the point of view of the observer co-moving with one of the non-inertial clocks. Let the position of the inertial clock be given by its co-ordinates  $(x, y)$ . We choose the origin of  $S$  such that, at  $t = t' = 0$ , the space origins of  $S$  and  $S'$  coincide and the velocity of the non-inertial observer is in the  $y$ -direction, as seen in  $S$ . The rate of clocks as seen by the non-inertial observer is given by

$$t = \gamma t' + \frac{\omega R}{c^2} [y \cos \gamma\omega t' - (x + R) \sin \gamma\omega t'] . \quad (20)$$

The oscillatory functions in (20) vanish when they are averaged over time. This means that the observer in  $S'$  agrees with the observer in  $S$  that the clock in  $S'$  is slower, but only in a time-averaged sense. For example, when these two clocks are very close to each other, then, by expanding (20) for small  $t'$ , one finds  $t = t'/\gamma$ , which is the result that one would obtain if the velocity of the non-inertial observer were constant. If the clock in  $S$  is in the centre, which corresponds to  $x = -R$ ,  $y = 0$ , then (20) gives  $t = \gamma t'$ , so in this case there is no oscillatory behaviour.

## 4 The invariance of radiation and of the velocity of light

In this section I study some physical effects for which an explicit transformation, such as (2), is not necessary. This allows to draw conclusions which refer to general (not only flat) space-time.

If (14) is interpreted as a proper frame of all observers on a rotating platform, then one can conclude that the observer on the rotating platform will see that the velocity of light depends on whether light is propagating in the clockwise or in the counter-clockwise direction (see, for example, [8]). This is related to the fact that the metric (15) is not time orthogonal. However, now we know that each observer belongs to a different Fermi frame, and from Property 2 we see that in the *vicinity* of any observer the metric is equal to the Minkowski metric  $\eta_{\mu\nu}$ . This implies that *for any local observer the velocity of light is isotropic and is equal to  $c$ , provided that it is measured by propagating a light beam in a **small** neighbourhood of the observer*. In particular, this leads to a slightly different interpretation of the Sagnac effect [3].

The result that (11) is given by (5) has the following very general implication. Let  $\Phi_{\alpha_1 \dots \alpha_n}(x)$  be an arbitrary local tensor quantity. Let the two observers measure this quantity at their common instantaneous position. The results of measurements will be related as

$$\Phi'_{\mu_1 \dots \mu_n} = f_{\mu_1}^{\alpha_1} \dots f_{\mu_n}^{\alpha_n} \Phi_{\alpha_1 \dots \alpha_n} . \quad (21)$$

The two measurements will be different if there is an instantaneous relative velocity between the two observers. However, the instantaneous relative acceleration, as well as higher-order derivatives, are irrelevant to this transformation law.

A local observer can measure only the values of fields at the point of his own position. It is completely unphysical to talk about the value of a field at some point from the point of view of the observer sitting at some other point. Since all interactions are local, a measuring apparatus can respond only to the values of fields at the position of the apparatus. This implies that *covariant fields seen by an observer depend only on his instantaneous velocity, not on his acceleration*.

This is in contrast with what was concluded in several previous papers (see references in [4]). In particular, it was concluded that an inertially moving charge radiated from the point of view of an accelerated observer and that the accelerated charge did not radiate from the point of view of a co-accelerating observer. These conclusions were an artifact of the misinterpretation of the Fermi co-ordinates of the accelerated observer, in the sense that they were applied to calculate how the fields transformed at *all* points covered by these co-ordinates, not only at the point of the position of the observer.

The use of Fermi co-ordinates also illuminates the origin of radiation of an accelerated charge. The radiation is not a kinematical effect resulting from the co-ordinate transformation between the frames of the radiating charge and the observer, but a dynamical effect, in the sense that even for the observer co-moving with the charge, the fields depend on acceleration. This is not explicitly seen in the conventional approach in which the Maxwell equations are solved in Minkowski co-ordinates. To see this explicitly, we write the covariant Maxwell equation

$$D_\mu F^{\mu\nu} = j^\nu \quad (22)$$

in a more explicit form

$$\partial_\mu F^{\mu\nu} + \Gamma_{\mu\lambda}^\mu F^{\lambda\nu} + \Gamma_{\mu\lambda}^\nu F^{\mu\lambda} = j^\nu . \quad (23)$$

We assume that the current  $j^\nu$  corresponds to a point-like charge. Let us study how the field  $F^{\mu\nu}$  looks like to an observer co-moving with the charge in his small neighbourhood. Since he uses the corresponding Fermi co-ordinates, the connections  $\Gamma_{\beta\gamma}^\alpha$  vanish in his small neighbourhood if and only if his trajectory is a geodesic (I assume that the charge does not rotate). Therefore, if the charge does not accelerate, in the small neighbourhood of the charge the solution of (23) looks just like the well-known Coulomb solution  $\mathbf{E} \propto r^{-2}$ ,  $\mathbf{B} = 0$ . On the other hand, if the charge accelerates, then, even in the small neighbourhood, Eqs. (23) no longer look like the Maxwell equations in Minkowski space-time. This gives rise to a more complicated solution, which includes the terms proportional to  $r^{-1}$ . The essential feature of radiating fields is the fact that they fall off with distance much slower than other fields, so their effect is much stronger at large distances. If the field is proportional to  $r^{-1}$  as seen by one observer, it is also so as seen by any other observer at the same position. This means that the notion of radiation does not depend on the observer.

As is well known, when a charge accelerates, a self-force acts on it. A self-force also appears when a charge moves geodesically in curved space-time [10, 11]. Fermi co-ordinates provide a simple intuitive picture explaining why a self-force appears when the charge accelerates or when space-time around the charge is curved. The fields produced by the charge always act on it. However, when the charge moves inertially through flat space-time, then the metric related to the corresponding Fermi co-ordinates is isotropic, and so are the fields. This implies that self-forces in different directions cancel exactly, so the resultant force is zero. When space-time is curved (such that it is not isotropic) or the charge accelerates, then the metric related to the Fermi co-ordinates is no longer isotropic. Consequently, the fields are also not isotropic, which implies that the resultant force need not to be zero. This implies that *a “freely” falling charge in curved space-time does not move along a geodesic and therefore radiates.*

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